Electrodynamics Ph.D. Qualifying Exam 2016/09/02 09:00–12:00

5 problems, total 100%

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1. (20 %) State and prove the uniqueness of the solutions of the Poisson equation with the Dirichlet and Neumann boundary conditions.

2. (20%)Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane $z = 0$ (and at infinity).

   (a) Write down the appropriate Green finction $G(\mathbf{x}, \mathbf{x}')$.

   (b) If the potential on the plane $z = 0$ is specified to be $\Phi = V$ inside a circle of radius a centered at the origin, and $\Phi = 0$ outside that circle, find an integral expression for the potential at the point P specified in terms of cylindrical coordinates (ρ, ϕ, z).

   (c) Show that, along the axis of the circle ($\rho = 0$), the potential is given by

   $$\Phi = V\left(1 - \frac{z}{\sqrt{a^2 + z^2}}\right)$$

   (d) Show that at large distances ($\rho^2 + z^2 \gg a^2$) the potential can be expanded in a power series in $(\rho^2 + z^2)^{-1}$, and that the leading terms are

   $$\Phi = \frac{Va^2}{2}\frac{z}{(\rho^2 + z^2)^{3/2}}\left[1 - \frac{3a^2}{4(\rho^2 + z^2)} + \frac{5(3\rho^2 a^2 + a^4)}{8(\rho^2 + z^2)^2} + \cdots\right]$$

   Verify that the results of (c) and (d) are consistent with each other in their common range of validity.

3. (20%) Consider a localized charge distribution $\rho(\mathbf{x})$ that gives rise to an electric field $\mathbf{E}(\mathbf{x})$ throughout space.

   (a) (8%)Show that the integral can be written as :

   $$\int_{r<R} \mathbf{E}(\mathbf{x}) d^3x = -\frac{R^2}{3\epsilon_0}\int d^3x' \frac{r_<}{r_>^2}\mathbf{n}'\rho(\mathbf{x}') \ .$$

   (b) (6%)Consider that the sphere of radius R completely encloses the charge density or the charge locates all exterior to the sphere of interest, separately. Verify that

   $$\int_{r<R} \mathbf{E}(\mathbf{x}) d^3x = -\frac{\mathbf{p}}{3\epsilon_0} \ ,$$
   $$\int_{r<R} \mathbf{E}(\mathbf{x}) d^3x = -\frac{4\pi}{3}R^3\mathbf{E}(0) \ .$$

(c) (6%)From the results of (b), show that the dipole field should be written as

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\mathbf{n}(\mathbf{p}\cdot\mathbf{n}) - \mathbf{p}}{|\mathbf{x}-\mathbf{x_0}|^3} - \frac{4\pi}{3}\mathbf{p}\delta(\mathbf{x}-\mathbf{x_0})\right].$$

4. (15%) Starting from the Biot and Savart law for a static magnetic field, derive the Ampere equation, $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$.
(Hint: $\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\times\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}d^3\vec{x}'$.)

5. (a) (10%) Starting from the Maxwell equation, show that the vector potential $\vec{A}(\vec{x},t)$ and scaler potential $\phi(\vec{x},t)$ satisfy the equations subject to the Lorentz gauge condition:

$$\begin{cases} \nabla^2\phi - \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2} = -\dfrac{\rho}{\epsilon_0}, \\ \nabla^2\vec{A} - \dfrac{1}{c^2}\dfrac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}. \end{cases}$$

(b) (10%) In the case of time harmonic source, ($\rho(\vec{x},t) = \rho(\vec{x})e^{-i\omega t}$, $\vec{J}(\vec{x},t) = \vec{J}(\vec{x})e^{-i\omega t}$,) show that the vector potential $\vec{A}(\vec{x},t)$ can be solved as

$$\vec{A}(\vec{x},t) = \vec{A}(\vec{x})e^{-i\omega t}, \text{ where } \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}d^3\vec{x}'.$$

Hint: You may directly use the expression of the Green's function,

$$G^{\pm}(\vec{x},t;\vec{x}',t') = \frac{\delta(t' - [t \mp \frac{|\vec{x}-\vec{x}'|}{c}])}{|\vec{x}-\vec{x}'|}.$$

(c) (5%) Derive the the expression for $\vec{A}(\vec{x})$ in the limit $kr \to \infty$.