1. Consider a one-dimensional (1D) simple harmonic oscillator (sho) of mass m and frequency ω which has the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \, ,$$

where $\hat{p}$ and $\hat{x}$ are, respectively, the momentum and the position operators.

(a) (10 points) If the lowering operator is defined as

$$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i\hat{p}}{m\omega} \right) \, ,$$

where $i \equiv \sqrt{-1}$. Express the Hamiltonian in terms of the lowering operator and its Hermitian conjugate.

(b) (10 points) If $|n\rangle$ is the n-th energy eigenstate of the 1D sho ($n = 0, 1, 2, \cdots$), show that the state

$$|\alpha\rangle \equiv e^{-\frac{\alpha}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \, ,$$

where α is a complex number, is an eigenstate for the lowering operator given in (a).

(c) (10 points) For any complex number α, show that the operator

$$\hat{D} \equiv \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a} \right)$$

can shift the lowering operator by α by way of a unitary transformation.

2. (20 points) Consider a pair of electrons constrained to move along the x-direction in a total spin triplet state. Suppose the electrons interact through an attractive potential

$$V(x_1, x_2) = \begin{cases} 0, & \text{if } |x_1 - x_2| > R, \\ -V_0, & \text{if } |x_1 - x_2| \leq R, \end{cases}$$

where V_0 and R are positive constants, and x_1, x_2 are coordinates of the two electrons. Find the eigen-energies of the two electrons when their total momentum is zero.

3. For time-independent perturbation theory, the Hamiltonian is $H = H^0 + H^1$ where H^1 is small compared to H^0. We assume that to every eigenket $|E_n^0\rangle \equiv |n^0\rangle$ of H^0 with eigenvalue E_n^0, and there is an eigenket $|n\rangle$ of H with eigenvalue E_n. The eigenkets and eigenvalues of H may be expanded in a perturbation series:

$$\begin{aligned} |n\rangle &= |n^0\rangle + |n^1\rangle + |n^2\rangle + \cdots \\ E_n &= E_n^0 + E_n^1 + E_n^2 + \cdots \end{aligned}$$

where the superscript k on each term gives the power of H^1.

(a) (20 points) Prove that $E_n^1 = \langle n^0|H^1|n^0\rangle$ and $E_n^2 = \langle n^0|H^1|n^1\rangle = \sum_m' \frac{|\langle n^0|H^1|m^0\rangle|^2}{E_n^0 - E_m^0}$, where $\sum_m'$ means that $m \neq n$.

(b) (10 points) Consider $H^1 = \lambda x^4$ for the oscillator problem. Show that $E_n^1 = \frac{3\hbar^2\lambda}{4m^2\omega^2}[1 + 2n + 2n^2]$.

4. In quantum mechanics, we have known the formula $J_\pm|jm\rangle = \hbar[(j \mp m)(j \pm m + 1)]^{1/2}|j, m \pm 1\rangle$ where $J_\pm = J_x \pm iJ_y$, $J^2|jm\rangle = j(j+1)\hbar^2|jm\rangle$, and $J_z|jm\rangle = m\hbar|jm\rangle$. Consider the problem of the two angular momenta $\mathbf{J}_1$ (system 1) and $\mathbf{J}_2$ (system 2) with $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ (the whole system).

(a) (10 points) Show that

$$|j_1 + j_2, j_1 + j_2 - 1\rangle = (\frac{j_1}{j_1 + j_2})^{1/2}|j_1(j_1 - 1), j_2 j_2\rangle + (\frac{j_2}{j_1 + j_2})^{1/2}|j_1 j_1, j_2(j_2 - 1)\rangle.$$

Note that the ket $|jm\rangle$ located in the lefthand side of "=" is the eigenket of the whole system, i.e., $j = j_1 + j_2$ and $m = j_1 + j_2 - 1$ in this case. Furthermore, the kets $|j_1 m_1, j_2 m_2\rangle \equiv |j_1 m_1\rangle \otimes |j_2 m_2\rangle$ located in the righthand side of "=" are the product kets of systems 1 and 2, e.g., the first term $|j_1 m_1, j_2 m_2\rangle \equiv |j_1 m_1\rangle \otimes |j_2 m_2\rangle = |j_1(j_1 - 1), j_2 j_2\rangle$ with $m_1 = j_1 - 1$ and $m_2 = j_2$.

(b) (10 points) Show that

$$|j_1 + j_2 - 1, j_1 + j_2 - 1\rangle = (\frac{j_1}{j_1 + j_2})^{1/2}|j_1 j_1, j_2(j_2 - 1)\rangle - (\frac{j_2}{j_1 + j_2})^{1/2}|j_1(j_1 - 1), j_2 j_2\rangle$$

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