Electrodynamics : Qualifying Exam. 2017/09/08 09:10–12:00

To get a full score, you have to finish problem 1,2,5 and one problem picked form problem 3 or 4. Total 100%

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1. (20%) Explicitly show that the electrostatic energy between two charges are the same for the formulism 1 and 2 up to some self-energy constant.
   Formulism 1:
   $$W_{int} = \frac{1}{4\pi\epsilon_0} \frac{q_1\, q_2}{|\vec{r_1} - \vec{r_2}|}$$
   Formulism 2:
   $$W_{int} = \frac{\epsilon_0}{2} \int |\vec{E}|^2 d^3x$$

2. (25%)Consider a localized charge distribution $\rho(\mathbf{x})$ that gives rise to an electric field $\mathbf{E}(\mathbf{x})$ throughout space.
   (a) (8%)Show that the integral can be written as :
   $$\int_{r<R} \mathbf{E}(\mathbf{x}) d^3x = -\frac{R^2}{3\epsilon_0} \int d^3x' \frac{r_<}{r_>^2} \mathbf{n}' \rho(\mathbf{x}') \ .$$
   (b) (9%)Consider that the sphere of radius R completely encloses the charge density or the charge locates all exterior to the sphere of interest, separately. Verify that
   $$\int_{r<R} \mathbf{E}(\mathbf{x}) d^3x = -\frac{\mathbf{p}}{3\epsilon_0} \ ,$$
   and
   $$\int_{r<R} \mathbf{E}(\mathbf{x}) d^3x = -\frac{4\pi}{3} R^3 \mathbf{E}(0) \ ,$$
   where $\mathbf{p}$ is the electric dipole moment.
   (c) (8%)From the results of (b), show that the dipole field should be written as
   $$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \left[\frac{3\mathbf{n}(\mathbf{p}\cdot\mathbf{n}) - \mathbf{p}}{|\mathbf{x} - \mathbf{x_0}|^3} - \frac{4\pi}{3} \mathbf{p} \delta(\mathbf{x} - \mathbf{x_0}) \right] \ .$$

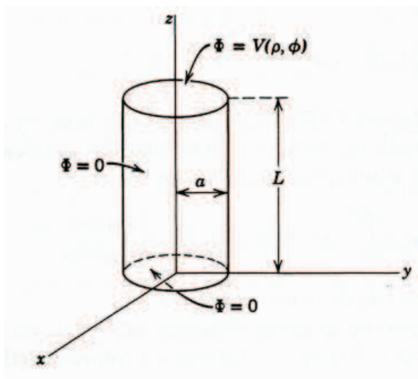
3. (25%) Consider a rectangular box with dimensions (a, b, c) in the (x, y, z) directions.
   (a) (7%)All surfaces of the box are kept at zero potential, except the surface $z = c$, which is at a potential $V(x, y)$. Find the potential everywhere inside the box.
   (b) (8%)Consider another case. The surface $z = c$ is at a potential $U_1(x, y)$, and the surface $y = b$ is at a potential $U_2(x, z)$. Other surfaces of the box are kept at zero potential. Find the potential everywhere inside the box.

(c) (10%)Derive the Green Function for this geometry as this form:

$$G(\mathbf{x},\mathbf{x}') = \frac{16\pi}{ab}\sum_{l,m=1}^{\infty} \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)$$
$$\times \frac{\sinh(K_{lm}z_<)\sinh[K_{lm}(c-z_>)]}{K_{lm}\sinh(K_{lm}c)}$$

where $K_{lm} = \pi(l^2/a^2 + m^2/b^2)^{1/2}$

4. (25%) Consider a cylinder which has a radius a and a hight L, the top and bottom surfaces being at $z = L$ and $z = 0$. The potential on the side and the bottom of the cylinder is zero, while the top has a potential $\Phi = V(\rho, \phi)$.



Please solve the Laplace equation in cylindrical coordinates to derive the general solution for the potential inside is

$$\Phi(\rho,\phi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho)\sinh(k_{mn}z)(A_{mn}\sin m\phi + B_{mn}\cos m\phi)$$

where $k_{mn} = \frac{x_{mn}}{a}$, and x_{mn} are the roots of $J_m(x_{mn}) = 0$. And find the expression for A_{mn} and B_{mn}.

5. (a) (7%) Starting from the Maxwell equation, show that the vector potential $\vec{A}(\vec{x},t)$ and scaler potential $\phi(\vec{x},t)$ satisfy the equations subject to the Lorentz gauge condition:

$$\begin{cases} \nabla^2\phi - \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2} = -\dfrac{\rho}{\epsilon_0}, \\ \nabla^2\vec{A} - \dfrac{1}{c^2}\dfrac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}. \end{cases}$$

(b) (8%) In the case of time harmonic source, ($\rho(\vec{x},t) = \rho(\vec{x})e^{-i\omega t}$, $\vec{J}(\vec{x},t) = \vec{J}(\vec{x})e^{-i\omega t}$,) show that the vector potential $\vec{A}(\vec{x},t)$ can be solved as

$$\vec{A}(\vec{x},t) = \vec{A}(\vec{x})e^{-i\omega t}, \text{ where } \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}d^3\vec{x}'.$$

Hint: You may directly use the expression of the Green's function,

$$G^{\pm}(\vec{x},t;\vec{x}',t') = \frac{\delta(t' - [t \mp \frac{|\vec{x}-\vec{x}'|}{c}])}{|\vec{x}-\vec{x}'|}.$$

(c) (7%) In the limit $kr \to \infty$, the expression for $\vec{A}(\vec{x})$ can be recast into

$$\lim_{kr\to\infty} \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}')e^{-ik(\vec{n}\cdot\vec{x}')}d^3\vec{x}' = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_n \frac{(-ik)^n}{n!}\int \vec{J}(\vec{x}')(\vec{n}\cdot\vec{x}')^n d^3\vec{x}',$$

where $\vec{n}$ is a unit vector in the direction of $\vec{x}$.

(d) (8%) By applying the result in (c) to the electric dipole radiation field, the electric and magnetic fields take the form in the radiation zone

$$\vec{H} = \frac{ck^2}{4\pi}(\vec{n}\times\vec{p})\frac{e^{ikr}}{r},$$

$$\vec{E} = \sqrt{\frac{\mu_0}{\epsilon_0}}\vec{H}\times\vec{n},$$

where $\vec{p} = \int \vec{x}'\rho(\vec{x}')d^3\vec{x}'$.