1. Two conducting planes, shown in Fig. 1, intersect at an angle β. The planes are assumed to be held at potential V near the origin where V is a constant. [Hint: in terms of the polar coordinates (ρ, ϕ), the Laplace equation in two dimensions is $\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\Phi}{\partial\rho})+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}=0$.]

   (a) (15%) Prove that the potential near the origin is approximately

$$\Phi(\rho, \phi) \approx V + a_1\rho^{\pi/\beta}\sin(\pi\phi/\beta),$$

   where a_1 is a constant.

   (b) (5%) Please write some sentences to describe how to obtain a_1.

2. Consider a point charge q embedded in a semi-infinite dielectric ϵ_1 a distance d away from a plane interface that separates the first medium from another semi-infinite dielectric ϵ_2. The surface may be taken as the plane $z = 0$, as shown in Fig. 2.

   (a) (5%) Consider that $\nabla \cdot \mathbf{E} = \alpha$ and $\nabla \times \mathbf{E} = \beta$. Determine α and β everywhere for this problem. [Please write down in SI units]

   (b) (5%) Determine the boundary condition at $z = 0$.

   (c) (15%) In attempting to use the image method, the potential at a point P described by cylindrical coordinates (ρ, ϕ, z) will be

$$\Phi = \begin{cases} \frac{1}{4\pi\epsilon_1}\left(\frac{q}{\sqrt{\rho^2+(d-z)^2}} + \frac{q'}{\sqrt{\rho^2+(d+z)^2}}\right), & z > 0 \\ \frac{1}{4\pi\epsilon_2}\left(\frac{q''}{\sqrt{\rho^2+(d-z)^2}}\right), & z < 0 \end{cases}$$

   where q' and q'' are two effective charges located at $(\rho' = 0, \phi', z' = -d)$ and $(\rho'' = 0, \phi'', z'' = d)$, respectively. Please use the result of (b) to determine q' and q'' in terms of q, ϵ_1, and ϵ_2.

3. (a) (5%) The basic differential laws of magnetostatics are $\nabla \cdot \mathbf{B} = \alpha$ and $\nabla \times \mathbf{B} = \beta$. Determine α and β in general. [Please write down in SI units]

   (b) (5%) Plot hysteresis loop [i.e., B or M (magnetization) as a function of H] of a ferromagnetic material.

   (c) (5%) Why do we need the gauge transform of the vector potential $\mathbf{A}$ for calculating $\mathbf{B}$?

4. (20%) The displacement $\vec{D}$ is related to the electric field, $\vec{E}$ by the relation : $\vec{D}(\vec{x},\omega) = \epsilon(\omega)\vec{E}(\vec{x},\omega)$, for the wave with the monochromatic components of frequency ω. The Fourier integral of the displacement in time time is expressed as:

$$\vec{D}(\vec{x},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{D}(\vec{x},\omega) e^{-i\omega t} d\omega.$$

(a) Show that the relation of $\vec{D}(\vec{x},t)$ and $\vec{E}(\vec{x},t)$ can be given as:

$$\vec{D}(\vec{x},t) = \epsilon_0 \{\vec{E}(\vec{x},t) + \int_{-\infty}^{\infty} K(\tau)\vec{E}(\vec{x},t-\tau)d\tau\},$$

where

$$K(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\epsilon(\omega)/\epsilon_0 - 1] e^{-i\omega\tau} d\omega.$$

(b) Consider a one-resonance version of the index of refraction :

$$\epsilon(\omega)/\epsilon_0 - 1 = \omega_p^2(\omega_0^2 - \omega^2 - i\gamma\omega)^{-1}.$$

Show that

$$K(\tau) = \omega_p^2 e^{-\gamma\tau/2} \frac{\sin \nu_0 \tau}{\nu_0} \theta(\tau),$$

where $\nu_0 = \sqrt{\omega_0^2 - \gamma^2/4}$, and $\theta(\tau)$ is the step function.

5. (a) (12%) Starting from the Maxwell equation, show that the vector potential $\vec{A}(\vec{x},t)$ and scaler potential $\phi(\vec{x},t)$ satisfy the equations subject to the Lorentz gauge condition:

$$\begin{cases} \nabla^2\phi - \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2} = -\dfrac{\rho}{\epsilon_0}, \\ \nabla^2\vec{A} - \dfrac{1}{c^2}\dfrac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}. \end{cases}$$

(b) (8%) In the case of time harmonic source, ($\rho(\vec{x},t) = \rho(\vec{x})e^{-i\omega t}$, $\vec{J}(\vec{x},t) = \vec{J}(\vec{x})e^{-i\omega t}$,) show that the vector potential $\vec{A}(\vec{x},t)$ can be solved as

$$\vec{A}(\vec{x},t) = \vec{A}(\vec{x})e^{-i\omega t}, \text{ where } \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} d^3\vec{x}'.$$

Hint: You may directly use the expression of the Green's function,

$$G^{\pm}(\vec{x},t;\vec{x}',t') = \frac{\delta(t' - [t \mp \frac{|\vec{x}-\vec{x}'|}{c}])}{|\vec{x}-\vec{x}'|}.$$