101 學年度

國立中正大學

博士班研究生資格考試

所別：物理研究所

科目：電　動　力　學

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101.09.07

1. (15%) Explicitly show that the electrostatic energy between two charges are the same for the formulism 1 and 2.
   Formulism 1:
$$W_{int} = \frac{1}{4\pi\epsilon_0}\frac{q_1\, q_2}{|\vec{r_1} - \vec{r_2}|}$$
   Formulism 2:
$$W_{int} = \frac{\epsilon_0}{2}\int |\vec{E}|^2 d^3x,$$
   where $\vec{E}$ is the electric field established by two point charges, q_1 and q_2, at positions $\vec{r_1}$ and $\vec{r_2}$, respectively.

2. (25%)Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane $z = 0$ (and at infinity).
   (a) Write down the appropriate Green finction $G(\mathbf{x}, \mathbf{x}')$.
   (b) If the potential on the plane $z = 0$ is specified to be $\Phi = V$ inside a circle of radius a centered at the origin, and $\Phi = 0$ outside that circle, find an integral expression for the potential at the point P specified in terms of cylindrical coordinates (ρ, ϕ, z).
   (c) Show that, along the axis of the circle ($\rho = 0$), the potential is given by
$$\Phi = V\left(1 - \frac{z}{\sqrt{a^2 + z^2}}\right)$$
   (d) Show that at large distances ($\rho^2 + z^2 \gg a^2$) the potential can be expanded in a power series in $(\rho^2 + z^2)^{-1}$, and that the leading terms are
$$\Phi = \frac{Va^2}{2}\frac{z}{(\rho^2 + z^2)^{3/2}}\left[1 - \frac{3a^2}{4(\rho^2 + z^2)} + \frac{5(3\rho^2 a^2 + a^4)}{8(\rho^2 + z^2)^2} + \cdots\right]$$
   Verify that the results of (c) and (d) are consistent with each other in their common range of validity.

3. (20%)
   (a) Construct the free-space Green function $G(x, y, x', y')$ for two-dimensional electrostatics by integrating $1/R$ with respect to $(z' - z)$ between the limits $\pm Z$, where Z is taken to be very large. Show that apart from an inessential constant, the Green function can be written alternately as
$$\begin{aligned} G(x, y; x'y') &= -\ln[(x - x')^2 + (y - y')^2] \\ &= -\ln[\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')] \end{aligned}$$
   (b) Show explicitly by separation of variables in polar coordinates that the Green function can be expressed as a Fourier series in' the azimuthal coordinate,
$$G = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi - \phi')} g_m(\rho, \rho')$$

where the radial Green functions satisfy

$$\frac{1}{\rho'}\frac{\partial}{\partial \rho'}(\rho'\frac{\partial g_m}{\partial \rho'}) - \frac{m^2}{\rho'^2}g_m = -4\pi\frac{\delta(\rho-\rho')}{\rho}$$

Note that $g_m(\rho, \rho')$ for fixed ρ is a different linear combination of the solutions of the homogeneous radial equation $\frac{\rho}{R}\frac{d}{d\rho}(\rho\frac{dR}{d\rho}) = n^2$ for $\rho' < \rho$ and for $\rho' > \rho$, with a discontinuity of slope at $\rho' = \rho$ determined by the source delta function.

(c) Complete the solution and show that the free-space Green function has the expansion

$$G(\rho, \phi; \rho', \phi') = -\ln(\rho_>^2) + 2\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho_<}{\rho_>})^m \cdot \cos[m(\phi - \phi')]$$

where $\rho_<(\rho_>)$ is the smaller (larger) of ρ and ρ'.

Some Useful Formulas.

Cylindrical (ρ, ϕ, z)

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

$$\int\frac{1}{\sqrt{a^2+x^2}}dx = \operatorname{arcsinh}(x/a) + c = \ln(x + \sqrt{x^2+a^2}) + c$$

4. (20%) (a) Show that the solution for the vector potential $\vec{A}(\vec{x},t)$ in the Lorentz gauge, where no boundary surfaces are present, is

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int d^3x' \int dt' \frac{\vec{J}(\vec{x}',t')}{|\vec{x}-\vec{x}'|}\delta(t' + \frac{|\vec{x}-\vec{x}'|}{c} - t).$$

(b) A localized system of charges and currents varies sinusoidally in time as the forms $\rho(\vec{x},t) = \rho(\vec{x})e^{-i\omega t}$ and $\vec{J}(\vec{x},t) = \vec{J}(\vec{x})e^{-i\omega t}$. Show that the solution for $\vec{A}$ becomes

$$\vec{A}(\vec{x}) = \frac{1}{c}\int \vec{J}(\vec{x}')\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}d^3x',$$

where $k = \omega/c$ is the wave number, and a sinusoidal time dependence is understood.

(c) In the far zone ($kr \gg 1$), show that

$$\vec{A}(\vec{x}) = \frac{e^{ikr}}{cr}\int \vec{J}(\vec{x}')e^{-ik\vec{n}\cdot\vec{x}'}d^3x',$$

where $\vec{n}$ is a unit vector in the direction of $\vec{x}$.

5. (20 %) Consider that the electric field of an electromagnetic wave in vacuum is as the following form:

$$E_x = 0,\ E_y = 0,\ E_z = 100\sin(\pi \times 10^7 t - \frac{\pi}{4}x),$$

where E is in volts/meter, t in seconds, and x in meters. Determine (a) the frequence f, (b) the wavelength λ, (c) the direction of propagation of the wave, (d) the direction of the magnetic field.