1. (15%) An energetic photon γ interacts with an electron with mass m at rest and produces an electron and n neutral particles M_i with same mass M, that is,

$$\gamma + e^- \to e^- + \sum_{i=1}^{n} M_i.$$

What is the threshold energy E_γ of the photon for this reaction?

2. (15%) A particle is in a three-dimensional box with $L_3 = L_2 = 2L_1$. Give the quantum numbers n_1, n_2, n_3 that correspond to the lowest six quantum states of this box.

3. A particle of mass m is confined in a one-dimensional box with potential energy:

$$V(x) = \begin{cases} \infty, & x < -a \\ 0, & -a < x < b \\ \infty, & b < x \end{cases},$$

where $a > 0$ and $b > 0$.

(a) (10%) Find the eigenfunctions and eigenenergies E_n.

(b) (10%) What is the first-order energy shift $\Delta E_n^{(1)}$ in the eigenstate n, if a small perturbation

$$\lambda H_1(x) = \begin{cases} 0, & -a < x < (b-a)/2 \\ \frac{\lambda}{a+b}(2x - b + a), & (b-a)/2 < x < b \end{cases}$$

is applied to the system? Here λ is a small real parameter.

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4. (30%) Consider an electron moving in an applied uniform magnetic field $\vec{B} = B\hat{z}$. The Hamiltonian of the electron will be

$$H = \frac{\vec{\Pi}^2}{2m},$$

where $\vec{\Pi} = -i\hbar\vec{\nabla} + \frac{e\vec{A}}{c}$ and $\vec{A} = B(-y/2, x/2, 0)$.

10% (a) Show that $[\Pi_x, \Pi_y] = -i\hbar m\Omega$, where $\Omega = \frac{eB}{mc}$.

10% (b) Let $a = \frac{1}{\sqrt{2m\hbar\Omega}}(\Pi_x - i\Pi_y)$. Show that $[a, a^+] = 1$ and $H = \hbar\Omega(a^+a + 1/2)$.

10% (c) Define $L_z = -i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$ and show that $[L_z, \Pi_x] = i\hbar\Pi_y, [L_z, \Pi_y] = -i\hbar\Pi_x$, and $[L_z, H] = 0$.

5. (20%) Consider a particle scattered by an one-dimensional potential well with an arbitrary shape. The particle wave function is $Ie^{ikx} + Oe^{-ikx}$ for the region $x \to -\infty$, and $I'e^{-ikx} + O'e^{+ikx}$ for the region $x \to \infty$. Let $\begin{pmatrix} O \\ O' \end{pmatrix} = S \begin{pmatrix} I \\ I' \end{pmatrix}$ where $S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$ and r, r', t and t' are complex numbers.

7% (a) By using probability-flux conservation show that S is unitary.

7% (b) Let $\begin{pmatrix} O' \\ I' \end{pmatrix} = M \begin{pmatrix} I \\ O \end{pmatrix}$. Express the 2x2 matrix M in terms of the variables: r, t, r', and t'.

6% (c) Make use of the probability-flux conservation to show that the matrix M will satisfy

$$M\Sigma_z M^+ = \Sigma_z,$$

where $\Sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.