1. (10 points)An antisymmetric square array B_{ij} (*i.e.*, $B_{ij} = -B_{ji}$) is given by

$$A_i = \frac{1}{2}\epsilon_{ijk}B_{jk} ,$$

where (A_1, A_2, A_3) is a pseudovector. (a) (5 points) Show that B_{jk} is a tensor. (b)(5 points) Show that $B_{ij} = \epsilon_{ijk}A_k$. (note that all the indices i, j, and k are from 1 to 3 and ϵ_{ijk} is the Levi-Civita symbol, which is a 3rd-rank pseudotensor.)

2. (10 points) Find a linear transform, describing a real rotation, connecting the real variables x, y, z and x', y', z' which is such that the quadratic form

$$F = 2xy + 2yz + 2zx ,$$

becomes a sum of squares

$$F = c_1(x')^2 + c_2(y')^2 + c_3(z')^2 .$$

Find c_1, c_2, c_3 and verify your transformation does indeed correspond to a real rotation.

3. (12 points) Find the general solutions of the following differential equations.

   (a) (6 points) $x\dfrac{dy}{dx} + y + x^4y^4e^x = 0$

   (b) (6 points) $y\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 - 6xy^2 = 0.$

4. (18 points) Consider a differential equation:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \alpha(\alpha+1)y = 0 .$$

   (a) (3 points) Find all singular points of the differential equation and determine whether each one is regular, regular singular or irregular singular.

   (b) (6 points) Find the indicial equations of the differential equation at $x = 0$ and $x = 1$. What are the leading exponents of the series solutions in each case?

   (c) (9 points) Near $x = 0$ and $\alpha = 4$, one of the two independent series solutions is a polynomial with finite powers. Find this solution.

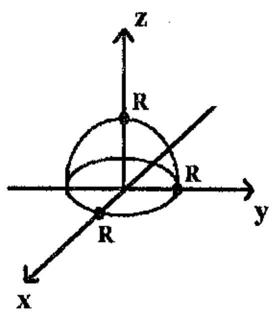
5. (a) Write down divergence theorem (3%)

(b) $\vec{V}=(r\cos\theta)\hat{r}+(r\sin\theta)\hat{\theta}+(r\sin\theta\cos\phi)\hat{\phi}$ Find $\nabla\bullet\vec{V}=?$ (5%)

(c) Check the divergence theorem for $\vec{V}$, using an inverted hemi-sphere as your volume (shown in below). Note the hemi-sphere is with radius R, and is resting on *xy* plane with its center on the origin. (12%)

You may need the following equation.

$$\nabla\bullet\vec{A}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta)+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$



6. $z\equiv x+iy$, prove the following equations

(a) $\cos z=\cos x\cosh y-i\sin x\sinh y$ (5%)

(b) $\sin z=\sin x\cosh y+i\cos x\sinh y$ (5%)

(c) $|\cos z|^2=\cos^2 x+\sinh^2 y$ (5%)

7. Using techniques of contour integrations, find

(a) $\int_0^{2\pi}\frac{d\theta}{\sqrt{2}-\cos\theta}=?$ (6%)

(b) $\int_0^{\infty}\frac{\sin x}{x}dx=?$ (9%)