1. (15 points) Consider an analytic function $f(x)$ which has the following Taylor expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n .$$

Suppose

$$\sum_{n=0}^{\infty} (\pm 1)^n \frac{f^{(n)}(0)}{n!} = \pm 1 ,$$

find $f(A)$, where $A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

2. (15 points) Consider a closed surface S enclosing a space region V. Suppose the vector fields $\mathbf{F}$ and $\mathbf{G}$ satisfy

$$\nabla \cdot \mathbf{F} = 0 \qquad \text{and} \qquad \mathbf{G} = \nabla \Phi ,$$

where Φ is a scalar function defined over S and V. If $\Phi = 0$ over S, find the volume integral

$$\int_V \mathbf{F} \cdot \mathbf{G} \, dV = ?$$

3. Using techniques of contour integrations, evaluate the following integrals:

(a) (10 points) $\displaystyle\int_0^{\infty} \frac{dx}{1+x^2}$.

(b) (10 points) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} \, dx$, where $0 < a < 1$.

4. Find the general solution of each of the following differential equations.

(a) (6 points) $y' + 2xy^2 = 0$.

(b) (6 points) $dx + (x - e^y)dy = 0$.

(c) (6 points) $xy' + y = e^{xy}$.

(d) (7 points) $ydy = (-x + \sqrt{x^2 + y^2})dx$.

5. (15 points) The exact equation of motion of a simple pendulum is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin\theta .$$

Integrate this equation once to find $d\theta/dt$ if $d\theta/dt = 0$ when $\theta = 90°$. Write a formula for $t(\theta)$ as an integral.

6. (10 points) Prove that if $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$, the the average of $[f(x)]^2$ over a period is $\sum_{-\infty}^{\infty} c_n c_{-n}$. What will be the average value for real $f(x)$?