1.(15%) The following matrix product is used in discussing a thick lens in air:

$$A = \begin{pmatrix} 1 & (n-1)/R_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d/n & 1 \end{pmatrix} \begin{pmatrix} 1 & -(n-1)/R_1 \\ 0 & 1 \end{pmatrix},$$

where d is the thickness of the lens, n is its index of refraction, and R_1 and R_2 are the radii of curvature of the lens surfaces. It can be shown that element A_{12} of A is $-1/f$, where f is the focal length of the lens.

(a) Evaluate A and det$A$.

(b) Find the focal length f.

2.(20%) The momentum p of an electron at velocity v near the velocity c of light increases according to the formula

$$p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}},$$

where m_0 is a constant (the rest mass). If an electron is subject to a constant force F, Newton's second law describing its motion is

$$\frac{dp}{dt} = \frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1 - v^2/c^2}}\right) = F.$$

(a) Find the velocity as a function of time and show that the limiting velocity as t tends to infinity is c.

(b) Find the distance traveled by the electron in time t if it starts from rest.

3.(15%) The force of gravitational attraction on a mass m a distance r from the center of the earth ($r >$ radius R of the earth) is mgR^2/r^2. Then the differential equation of motion of a mass m projected radially outward from the surface of the earth, with initial velocity v_0, is

$$m\frac{d^2 r}{dt^2} = -mg\frac{R^2}{r^2}.$$

(a) Find the velocity v as a function of r if $v = v_0$ initially (that is, when $r = R$).

(b) Find the maximum value of r for a given v_0, that is, the value of r when $v = 0$.

(c) Find the escape velocity, that is, the smallest value of v_0 for which r can tend to infinity.

4. (15%) Evaluate the following two integrals using the residue theorem.

(a) (7%) $\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$,

(b) (8%) $\int_0^{\infty} \frac{x^2 dx}{\left(x^2+4\right)\left(x^2+9\right)}$.

5. (20%) Let $\{\psi_n(x)\}$ be a set of orthogonal and complete basis functions for the range (x_1, x_2). Then a well behaved function $f(x)$ can be expanded as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \psi_n(x)$$

in this range with

$$c_n = \frac{\int_{x_1}^{x_2} f(x)\psi_n^*(x)dx}{\int_{x_1}^{x_2} |\psi_n(x)|^2 dx}.$$

(a) (10%) Given $\psi_n(x) = e^{i2\pi n v_0 x}$, determine the appropriate v_0's to make $\{\psi_n(x)\}$ orthogonal and complete for $(x_1, x_2) = (-1,1)$ and $(-2,2)$ respectively.

(b) (10%) After v_0's are appropriately determined in (a), we can express

$f(x) = \begin{cases} e^{i2\pi x} & \text{for } -1<x<1 \\ 0 & \text{otherwise} \end{cases}$ as $\sum_{n=-\infty}^{\infty} c_n e^{i2\pi n v_0 x}$ in the ranges $(-1,1)$ and $(-2,2)$ respectively. Please determine the corresponding c_n's.

6. (15%) Let $\phi = \sin\left[\left(\vec{a}\cdot\vec{r}\right)\left(\vec{b}\cdot\vec{r}\right)\right]$ and $\vec{A} = x^2y^2\hat{x} + 2x^3y\hat{y}$, where $\vec{a}$ and $\vec{b}$ are constant vectors as well as $\hat{x}$ and $\hat{y}$ are unit vectors along the perpendicular axes of a two dimensional Cartesian coordinate.

(a) (7%) derive $\nabla\phi$ (hint: $d\phi = \nabla\phi\cdot d\vec{r}$),

(b) (8%) Is the integral $\int_{x=0,y=0}^{x=1,y=1} \vec{A}\cdot d\vec{r}$ path dependent? If so, calculate the integral along the following two paths: (i) $(0,0)\to(1,0)\to(1,1)$ and (ii) $(0,0)\to(1,1)$. If not, do your calculation along any single path.