

- 1.(10%) Find a continuous solution satisfying

$$\frac{dy}{dx} + y = f(x) \quad \text{where} \quad f(x) = \begin{cases} 1, & 0 \leq x \\ 0, & x > 1 \end{cases}$$

and the initial condition $y(0) = 0$.

- 2.(15%) Use the power series method to solve the following differential equation

$$y'' + (\cos x)y = 0.$$

- 3.(15%) Evaluate the following integral

$$\int_0^{\infty} \frac{\ln x}{x^2 + 4} dx.$$

- 4.(10%) Let X and Y be independent Poisson variables with parameters λ and μ . Show that the sum $X + Y$ is also Poisson.

5. (10%) Two lines have direction cosines $\cos \alpha, \cos \beta, \cos \gamma$, and $\cos \alpha', \cos \beta', \cos \gamma'$, respectively. Form trigonometric considerations, prove that the angle θ between these lines is given by

$$\cos \theta = \cos \alpha \cos \alpha' + \cos \beta \cos \beta' + \cos \gamma \cos \gamma'.$$

6. (10%) Show that the following matrix is a unitary matrix.

$$\begin{pmatrix} \frac{1}{4}(1 + i\sqrt{3}) & \frac{\sqrt{3}}{2\sqrt{2}}(1 + i) \\ \frac{-\sqrt{3}}{2\sqrt{2}}(1 + i) & \frac{1}{4}(\sqrt{3} + i) \end{pmatrix}$$

7. (15%) Given the function $f(x) = |x|$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the Fourier series for $f(x)$.

8. (15%) We consider an example of the use of the diagonalization process. A quadratic surface has the equation

$$x^2 + 6xy - 2y^2 - 2yz + z^2 = 24.$$

Rotate this quadratic surface to the principal axes.