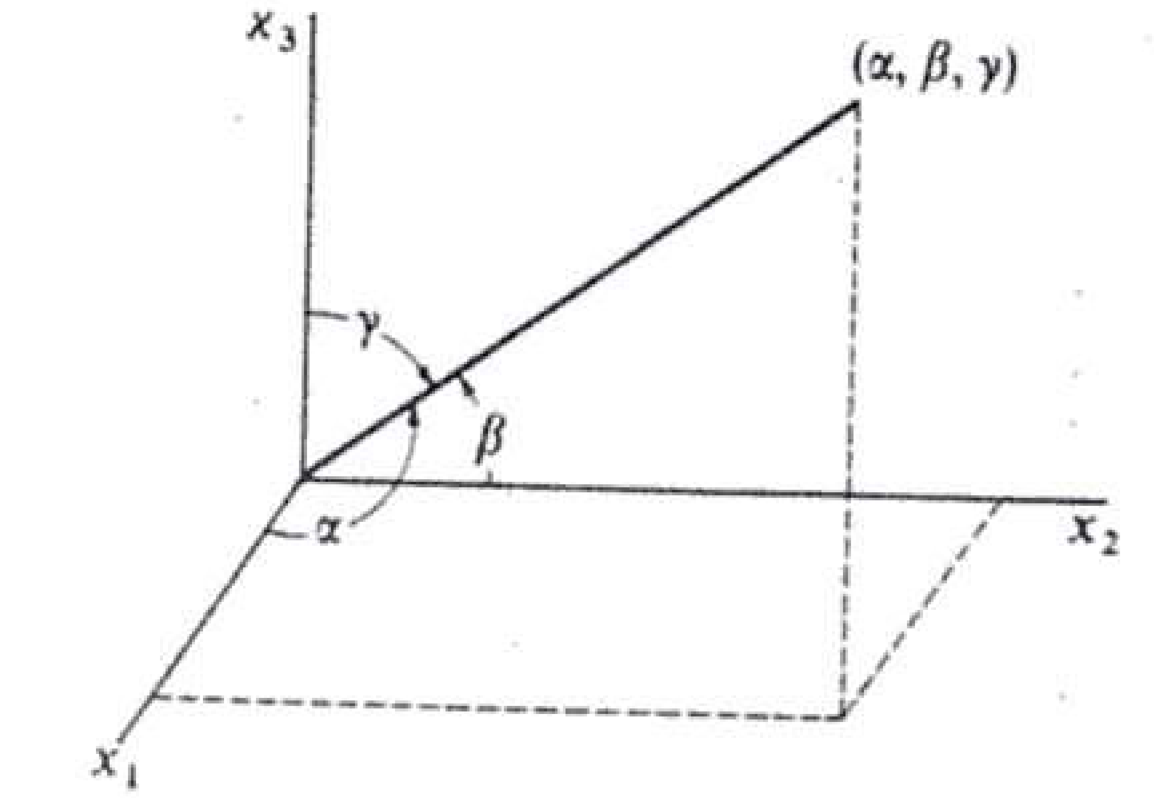
1. (10 points) Consider the figure shown below. Prove the direction cosine equation from trigonometric considerations

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$




2. (20 points) Find the characteristic frequencies for the case of the two masses ($m_1 = m_2 = M$) connected by a spring to each other and by springs to fixed positions (Figure 12-1).

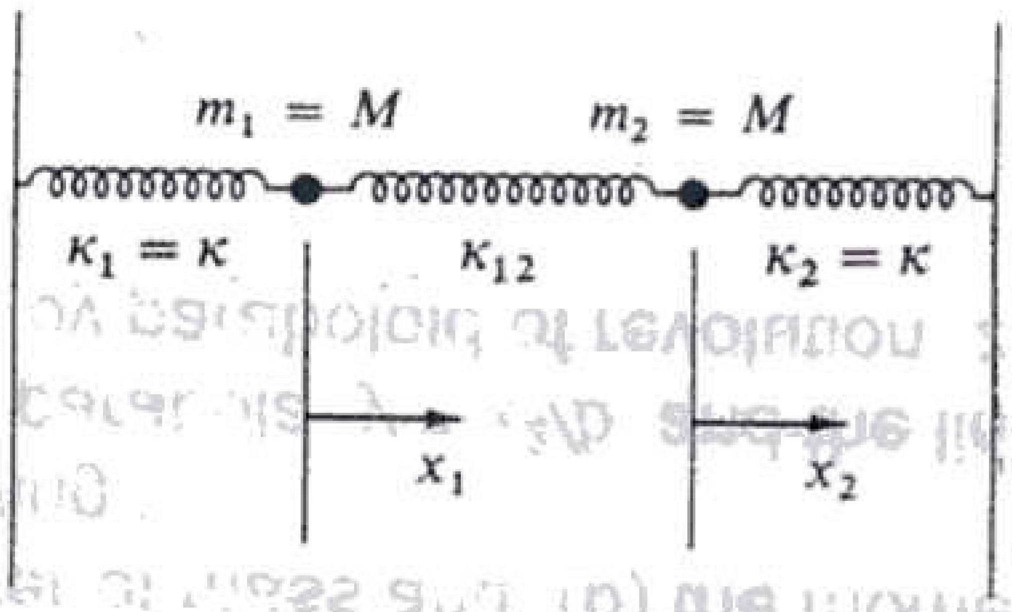



FIGURE 12-1

3. (20 points) Given a Lagrangian function $L(q_k, \dot{q}_k; t)$, the Hamiltonian function is then defined by

$$H(q_k, p_k; t) = \sum_j p_j \dot{q}_j - L(q_k, \dot{q}_k; t),$$

where $p_k \equiv \partial L / \partial \dot{q}_k$ is the momentum conjugated to the generalized coordinate $q_k$. Show by direct computation that the Hamilton's equations of motion

$$\dot{q}_k = \frac{\partial H}{\partial p_k},$$
$$-\dot{p}_k = \frac{\partial H}{\partial q_k}.$$

(*Hint*: Use the Lagrange's equation of motion $\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0$.)

4. (20 points) A $2\ Kg$ weight hung on a vertical spring stretches it $0.4\ m$. The weight is then pulled down $1\ m$ and released. (a) Find the position of the weight at any time if a damping force numerically equal to $15$ times the instantaneous speed is acting. (b) Is the motion oscillatory damped, overdamped or critically damped ?
(Use $g = 10\ m/s^2$)

5. (10 points) (a) Show that $\mathbf{F} = (6xy + z^3)\,\mathbf{i} + 3x^2\,\mathbf{j} + 3xz^2\,\mathbf{k}$ is a conservative force field. (b) Find the potential. (c ) Find the work done in moving an object in this field from $(1,-3,2)$ to $(3,2,-1)$.

6. (20 points) Find (a) the center of mass and (b) the moment of intertia about its symmetry axes of the following :
   (i) The area bounded by parabola $y = x^2/b$ and the line $y = b$.
   (ii) The volume bounded by paraboloid of revolution $z = (x^2 + y^2)/b$ and the plane $z = b$.