Qualify Exam–QM (2009)

1. For a state that has the momentum-space wave function

$$\phi(p) = (2\pi\sigma^2)^{-1/4} \exp\left(-\frac{p^2}{4\sigma^2}\right),$$

where σ is a positive real number:

(a) (10 points) Check whether $\phi(p)$ is a "minimum uncertainty" state, *ie*

$$\Delta\hat{p}^2\,\Delta\hat{x}^2 = \frac{\hbar^2}{4}$$

with $\Delta\hat{p}^2 \equiv \langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2$ and likewise for $\Delta\hat{x}^2$.

(b) (10 points) If $\phi(p)$ is the ground state of a one-dimensonal simple harmonic oscillator (1D sho) which has the Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2,$$

determine the value of σ.

(c) (10 points) If in $\phi(p)$ we have σ *less than* the value found in (b), find the probability that the state would have an odd number of 1D sho energy quanta. In other words, find the overlap between $\phi(p)$ and the 1D sho enegy eigenstates

$$\psi_n(x) = \frac{1}{\sqrt{2^n\,n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(y - \frac{d}{dy}\right)^n e^{-y^2/2}\Bigg|_{y=\sqrt{\frac{m\omega}{\hbar}}x}$$

for all $n = 1, 3, 5, \ldots$.

2. (20points) Consider two identical spin-1/2 particles with the Hamiltonian

$$H = g\,\hat{S}_1^z\,\hat{S}_2^x,$$

where $\hat{S}_i^\alpha$ is the spin operator for particle i in α direction, g is the coupling constant. If at time $t = 0$ particle 2 is prepared in the state

$$|\chi_2\rangle = \frac{1}{\sqrt{2}}\left(i|\uparrow\rangle + |\downarrow\rangle\right),$$

find the state of the total system at time $t = \pi/(g\hbar)$ if at $t = 0$ particle 1 starts from the state

$$|\chi_1\rangle = a|\uparrow\rangle + b|\downarrow\rangle,$$

where a, b are real constants that satisfy $a^2 + b^2 = 1$. You may need the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

3. (a)(10points) Let E_0 be the lowest eigenvalue of the hamiltonian H. Show that

$$\frac{< \Psi|H|\Psi >}{< \Psi|\Psi >} \geq E_0$$

where $|\Psi >$ is an arbitrary state.

(b) (10points) Estimate the ground state energy of a particle with mass m in a potential $V(x) = \lambda x^4$ by using the trial fnction $\Psi_\alpha(x) = e^{-\alpha x^2/2}$ where α is a free parameter.

4. (10points) Solving the possible bound-state energy of the one-dimensional potential problem with $V(x) = -aV_0\delta(x)$ for a spinless particle with mass m.

5. (20points) Consider a spinless particle with mass m moving in a infinite potential well

$$V(x) = 0, \quad 0 \leq x \leq 2L; \qquad V(x) = \infty, \quad otherwise.$$

If $V(x)$ is modified by the potential $V_p(x) = \lambda V_0 \delta(x - L)$ where $\lambda \ll 1$, then using first-order perturbation theory to calculate the energy of the nth excited state