1. The Schwinger's oscillator model of angular momentum. Let us consider two uncoupled simple harmonic oscillators, which we call the plus type and the minus type. We have the annihilation and creation operators, denoted by a_+ and $a_+^\dagger$ for the plus-type oscillator; likewise, we have a_- and $a_-^\dagger$ for the minus-type oscillator. We define the number operators N_+ and N_- as follows:

$$N_+ \equiv a_+^\dagger a_+ , \qquad N_- \equiv a_-^\dagger a_- .$$

We assume that the usual commutation relations among a, $a^\dagger$, and N hold for oscillators of the same type

$$\begin{aligned} [a_+, a_+^\dagger] &= 1, & [a_-, a_-^\dagger] &= 1, \\ [N_+, a_+] &= -a_+, & [N_-, a_-] &= -a_-, \\ [N_+, a_+^\dagger] &= a_+^\dagger, & [N_-, a_-^\dagger] &= a_-^\dagger. \end{aligned} \tag{1}$$

However, we assume that any pair of operators between different oscillators commute:

$$[a_+, a_-^\dagger] = 0, \qquad [a_-, a_+^\dagger] = 0,$$

and so forth. Because the two oscillators are uncoupled and N_+ and N_- commute, we can build up simultaneous eigenkets of N_+ and N_- with eigenvalues n_+ and n_-, respectively. That is

$$N_+ |n_+, n_-\rangle = n_+ |n_+, n_-\rangle , \qquad N_- |n_+, n_-\rangle = n_- |n_+, n_-\rangle ,$$

where the vacuum ket is defined by

$$a_+ |0, 0\rangle = 0, \qquad a_- |0, 0\rangle = 0.$$

a. (5 points) Find the general expression of the eigenket $|n_+, n_-\rangle$.

b. (10 points) Next, we define

$$J_+ \equiv \hbar a_+^\dagger a_-, \qquad J_- \equiv \hbar a_-^\dagger a_+,$$

and

$$J_z \equiv \frac{\hbar}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right) = \frac{\hbar}{2}\left(N_+ - N_-\right).$$

Prove that these operators satisfy the angular-momentum commutation relations of the usual form.

c. (5 points) What is the relation between the two sets of quantum numbers (n_+, n_-) of the oscillator model and (j, m) of the familiar angular-momentum system. Express the eigenket $|n_+, n_-\rangle$ in terms of j and m.

2. Consider a three-dimensional ket space. If a certain set of orthonormal kets – say, $|1\rangle$, $|2\rangle$, and $|3\rangle$ – are used as the base kets, the operators A and B are represented by

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \qquad B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

with a and b both real.

a. (4 points) Obviously A exhibits a degenerate spectrum. Does B also exhibit a degenerate spectrum?

b. (4 points) Show that A and B commute.

c. (7 points) Find a new set of orthonormal kets which are simultaneous eigenkets of both A and B. Specify the eigenvalues of A and B for each of the three eigenkets. Does your specification of eigenvalues completely characterize each eigenket?

3. Consider an electron confined to the interior of a hollow cylindrical shell whose axis coincides with the z-axis. The wave function is required to vanish on the inner and outer walls, $\rho = \rho_a$ and ρ_b, and also at the top and bottom, $z = 0$ and L.

   a. (5 points) Find the energy eigenfunctions. (Do not bother with normalization.) Show that the energy eigenvalues are given by

$$E_{lmn} = \frac{\hbar^2}{2m_e}\left[k_{mn}^2 + \left(\frac{l\pi}{L}\right)^2\right], \qquad (l = 1, 2, 3, ..., m = 0, 1, 2, ...),$$

   where k_{mn} is the nth root of the transcendental equation

$$J_m(k_{mn}\rho_b)N_m(k_{mn}\rho_a) - N_m(k_{mn}\rho_b)J_m(k_{mn}\rho_a) = 0.$$

   b. (5 points) Repeat the same problem where there is a uniform magnetic field $\vec{B} = B\hat{z}$ for $0 < \rho < \rho_a$. Note that the energy eigenvalues are influenced by the magnetic field even though the electron never "touches" the magnetic field.

   c. (5 points) Compare, in particular, the ground state of the $\vec{B} = 0$ problem with that of the $\vec{B} \neq 0$ problem. Show that if we require the ground-state energy to be unchanged in the presence of B, we obtain "flux quantization"

$$\pi\rho_a^2 B = \frac{2\pi N\hbar c}{e}, \qquad (N = 0, \pm 1, \pm 2, ...).$$

(*Hints:*

1. The gradient and Laplacian operators in the cylindrical coordinates are, respectively,

$$\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z},$$
$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}.$$

2. The Bessel's differential equation is defined by

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0.$$

   The most general solution for any ν can be written as

$$y_\nu(x) = AJ_\nu(x) + BN_\nu(x),$$

   where A and B are constants.

3. The Schrodinger equation in the presence of electromagnetic field is given by

$$\left[\frac{1}{2m_e}\left(\vec{p} - \frac{e\vec{A}(\vec{x})}{c}\right)^2 + e\phi(\vec{x})\right]\psi(\vec{x}) = E\psi(\vec{x}),$$

   where $\phi(\vec{x})$ and $\vec{A}(\vec{x})$ are scalar and vector potentials, respectively.)

4. **a**. (10 points) Two spin-0 identical Bosons are confined in the one-dimensional potential well

$$V(x) = \begin{cases} 0 & |x| < L/2\,, \\ \infty & |x| > L/2\,. \end{cases}$$

Find the ground state wave function for the system.

b. (10 points) Consider the same problem as (a) but now with two spin-$\frac{1}{2}$ identical Fermions.

5. The Hamiltonian for a rigid rotator in a transverse magnetic field takes the form

$$H = A\mathbf{L}^2 + BL_z + CL_y\,,$$

where $\mathbf{L}$ is the operator for orbital angular momentum and A, B, C, are real constants.

a. (10 points) If $B \gg C$, find the energy eigenvalue to the lowest non-vanishing order in perturbation.

b. (10 points) For the matrix element $\langle n', l', m'|(3z^2 - y)|n, l, m\rangle$, what are the selection rules?

6. (20 points) Consider an electron-positron composite system in a uniform, static magnetic field $\mathbf{B}$. The Hamiltonian for the system is

$$H = A\,\mathbf{S}_1 \cdot \mathbf{S}_2 + \frac{eB}{mc}\,(S_{1z} - S_{2z})\,,$$

where A is a real constant, $\mathbf{S}_{1,2}$ are, respectively, the spin operators for the electron and the positron, c is the speed of light, $e < 0$ is the electron charge, and m is the mass of the electron. If $B \ll |mcA/e|$, find the energy eigenvalues of the system to lowest non-vanishing order in perturbation theory.