Qualify Exam.(Q.M./2006/Spring)

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1. (20 points) Consider a particle with mass M confined in a three-dimensional cubic box of volume L^3 ($V = 0$ for $0 \leq x, y, z \leq L$, and $V = \infty$ otherwise). (a) Find the normalized eigen-functions (b) Find the first 3 energy levels and discuss their degeneracy.

2. (20 points) Let $H = a^\dagger a + 1/2$ be the hamiltonian of a quantum harmonic oscillator with eigenstates $|n\rangle$ satisfying

$$H|n\rangle = (n + 1/2)|n\rangle, \quad a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad n \geq 0$$

where $[a, a^\dagger] = 1$ and we set the Planck's constant $\hbar$ and the frequency of the oscillator ω equal to one. The coherent state $|\lambda\rangle$ is defined by $a|\lambda\rangle = \lambda|\lambda\rangle$. Show that

$$|\lambda\rangle = e^{\lambda a^\dagger}|0\rangle.$$

[Hint: Let $|\lambda\rangle = \sum_{n=0}^{\infty} f_n|n\rangle$]

3. (20 points) Let H be the hamiltonian of a system. If $E[\Psi]$ stands for the mean value of the energy in the state $|\Psi\rangle$, i.e.

$$E[\Psi] = \frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}.$$

Show that $E[\Psi] \geq E_0$ where E_0 is the lowest eigenvalue of H.

4. (20 points) Consider a system with two spin-$\frac{1}{2}$ particles which has the Hamiltonian

$$H(t) = \begin{cases} 0 & \text{for } t < 0, \\ \gamma\left(\vec{S}_1 \cdot \vec{S}_2\right) & \text{for } t > 0, \end{cases}$$

where γ is a number and $\vec{S}_{1,2}$ are the spin operators for the particles. If at $t = 0$ the system is in the state $|\uparrow\rangle \otimes |\downarrow\rangle \equiv |\uparrow\downarrow\rangle$, find the probability that the system will be found in the state $|\downarrow\uparrow\rangle$ at any time $t > 0$.

5. (20 points) A particle of mass m and charge $q > 0$ is subject to the electric field

$$E(x) = \begin{cases} -E_0 & \text{for } x < 0, \\ E_0 & \text{for } x > 0, \end{cases}$$

Estimate the ground state energy of the particle using the WKB method.

6. (20 points) Consider a hydrogen atom originally in its ground state at time $t = 0$. If it is under the action of a uniform electric field $\hat{z}\,E_0 \exp(-t/\tau)$, where E_0 and τ are constants. Suppose E_0 is small, find the transition probability of the atom to the state $|n = 2, l = 1, m\rangle$. You may need the matrix element $\langle 21m|z|100\rangle = a_0\beta\delta_{m0}$, where a_0 is the Bohr radius and β is a constant. The energy difference between $n = 1$ and $n = 2$ states of the hydrogen atom is $3e^2/(8a_0)$.