101 學年度

國立中正大學

博士班研究生資格考試

所別：物理研究所

科目：量　子　力　學

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101.09.14

QM–part I

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1. (15 points) If a beam of photons are prepared in a polarized state, which is known to be either of

$$|\epsilon_1\rangle = \cos\theta|e_1\rangle + \sin\theta|e_2\rangle \quad \text{or} \quad |\epsilon_2\rangle = \cos\theta|e_1\rangle + e^{i\phi}\sin\theta|e_2\rangle\,,$$

where the basis states $|e_{1,2}\rangle$ are, respectively, the polarized states along space directions $\hat{e}_{1,2}$. Is it possible to determine the polarization state of the beam with just one single polaroid (with many measurements if necessary)? Explain the reason for your answer as clearly as possible.

2. (15 points) A charged particle with charge q is moving on the xy-plane in the presence a uniform magnetic field $\vec{B} = B\hat{z}$. Ignoring the spin of the particle, find the eigenenergies of the particle. You may use the vector potential $\vec{A} = -By\,\hat{x}$ if you wish.

3. (20 points) Consider a quantum system with two (orthonormal) states $|1\rangle$, $|2\rangle$ whose Hamiltonian is

$$H = \hbar\omega\,(\,i|2\rangle\langle 1| - i|1\rangle\langle 2|\,)\,,$$

where $\omega > 0$ is a constant and $i \equiv \sqrt{-1}$. If at time $t = 0$ the observable

$$O = |1\rangle\langle 1| - 2|2\rangle\langle 2|$$

has expectation value -2, for any time $t > 0$ find the probability that the system would be found in state $|2\rangle$.

QM–part II

4. (10 points) Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two spin angular momenta. Show that $\mathbf{P}_1 = \frac{3}{4}\mathbf{I} + (\mathbf{S}_1 \cdot \mathbf{S}_2)/\hbar^2$ and $\mathbf{P}_0 = \frac{1}{4}\mathbf{I} - (\mathbf{S}_1 \cdot \mathbf{S}_2)/\hbar^2$ are projection operators, that is, obey $\mathbf{P}_i\mathbf{P}_j = \delta_{ij}\mathbf{P}_j$. Here, $\mathbf{I}$ is the 2×2 identity matrix.

5. (25 points) Consider an electron bound to a proton in a state of orbital angular moment l. Since the electron has spin 1/2, its total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ can have values of $j = l \pm 1/2$. We wish to express the total-j states in terms of the product states $|l, m_l; s, m_s\rangle$, in which m_l and m_s stand for orbital and spin angular momentum projections along the z axis, respectively. Since $m_s = \pm 1/2$, at each m there will be at the most two eligible product kets. Let

$$\begin{aligned} |j = l + 1/2, m\rangle &= \alpha\,|l, m - 1/2; 1/2, 1/2\rangle + \beta\,|l, m + 1/2; 1/2, -1/2\rangle\,, \\ |j = l - 1/2, m\rangle &= \alpha'\,|l, m - 1/2; 1/2, 1/2\rangle + \beta'\,|l, m + 1/2; 1/2, -1/2\rangle\,, \end{aligned}$$

where m stands for total angular momentum projection along the z axis.

(a) (5 points) Express the total angular momentum square J^2 in terms of L^2, S^2, L_z, S_z, $L_\pm$, and $S_\pm$. Here, $L_\pm = L_x \pm iL_y$ and likewise $S_\pm = S_x \pm iS_y$.

(b) (10 points) Use the result of part (a) and the requirement that the above ket-states are orthonormal to deduce the ratio of coefficients α/β.

(c) (10 points) From the result of part (b), you find all the coefficients α, β, α', and β'. Determine explicitly the states $|j = l \pm 1/2, m\rangle$ in terms of $|l, m - 1/2; 1/2, 1/2\rangle$ and $|l, m + 1/2; 1/2, -1/2\rangle$, up to the overall sign.

6. (a) (10 points) Prove the Thomas-Reiche-Kuhn sum rule

$$\sum_{n'} (E_{n'} - E_n)\,\left|\langle n'|\hat{x}|n\rangle\right|^2 = \frac{\hbar^2}{2m}\,,$$

where $|n\rangle$ and $|n'\rangle$ are eigenstates of the Hamiltonian: $H = \hat{p}^2/2m + V(\hat{x})$.

(b) (5 points) Test the sum rule on the nth state of the harmonic oscillator, where $H = \hat{p}^2/2m + m\omega^2\hat{x}^2/2$. Note that $A = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\frac{\hat{p}}{\sqrt{2m\omega\hbar}}$ and $A^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - i\frac{\hat{p}}{\sqrt{2m\omega\hbar}}$, with $A\,|n\rangle = \sqrt{n}\,|n-1\rangle$ and $A^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$.