1 (15%) Show that the interaction potential energy of two point charges q_1 and q_2 located at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is given by

$$W_{\text{int}} = \frac{q_1 q_2}{16\pi^2\epsilon_0} \int \frac{(\boldsymbol{x}-\boldsymbol{x}_1)\cdot(\boldsymbol{x}-\boldsymbol{x}_2)}{|\boldsymbol{x}-\boldsymbol{x}_1|^3|\boldsymbol{x}-\boldsymbol{x}_2|^3} d^3x$$

Hints: Perform a change of integration variable to $\boldsymbol{\rho} = (\boldsymbol{x}-\boldsymbol{x}_1)/|\boldsymbol{x}_1-\boldsymbol{x}_2|$

2 (15%) Find the Green function for a Dirichlet problem inside a rectangular box defined by the six planes, $x = 0$, $y = 0$, $z = 0$, $x = a$, $y = b$, $z = c$. The expansion is to be made in terms of eigenfunctions of the wave equation

$$(\nabla^2 + k^2)\psi(x, y, z) = 0 .$$

3 (20%) Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane $z = 0$ (and at infinity)

(a) Write down the appropriate Green function $G(\boldsymbol{x}, \boldsymbol{x}')$

(b) If the potential on the plane $z = 0$ is specified to be $\Phi = V$ inside a circle of radius a centered at the origin, and $\Phi = 0$ outside the circle, find an integral expression for the potential at the point P specified in terms of cylindrical coordinates (ρ, ϕ, z)

(c) Show that, along the axis of the circle ($\rho = 0$), the potential is given by

$$\Phi = V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

(d) Show that at large distances ($\rho^2 + z^2 \gg a^2$) the potential can be expanded in a power series in $(\rho^2 + z^2)^{-1}$, and that the leading terms are

$$\Phi = \frac{Va^2}{2}\frac{z}{(\rho^2+z^2)^{3/2}}\left[1 - \frac{3a^2}{4(\rho^2+z^2)} + \frac{5a^2(3\rho^2+a^2)}{8(\rho^2+z^2)^2} + \cdots\right]$$

Verify that the results of parts (c) and (d) are consistent with each other in their common range of validity

Note that the Laplacian of a scalar function ψ in cylindrical coordinates (ρ, ϕ, z) is

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} .$$

4. (15%) From the macroscopic point of view, the electric potential at some point in the medium can be built up by linear superposition of the contributions from each macroscopically small volume element ΔV at the variable point $\vec{x}'$. Show that the first Maxwell equation macroscopically can be written as

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\left[\rho - \vec{\nabla}\cdot\vec{P}\right],$$

where $\vec{P}$ is the dipole momentum per unit volume

5. (a) (7%) Starting from the Maxwell equation, show that the vector potential $\vec{A}(\vec{x},t)$ and scaler potential $\phi(\vec{x},t)$ satisfy the equations subject to the Lorentz gauge condition:

$$\begin{cases} \nabla^2\phi - \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2} = -\dfrac{\rho}{\epsilon_0}, \\ \nabla^2\vec{A} - \dfrac{1}{c^2}\dfrac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} \end{cases}$$

(b) (8%) In the case of time harmonic source, ($\rho(\vec{x},t) = \rho(\vec{x})e^{-i\omega t}$, $\vec{J}(\vec{x},t) = \vec{J}(\vec{x})e^{-i\omega t}$,) show that the vector potential $\vec{A}(\vec{x},t)$ can be solved as

$$\vec{A}(\vec{x},t) = \vec{A}(\vec{x})e^{-i\omega t}, \text{ where } \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}d^3\vec{x}'$$

Hint: You may directly use the expression of the Green's function,

$$G^{\pm}(\vec{x},t;\vec{x}',t') = \frac{\delta(t' - [t \mp \frac{|\vec{x}-\vec{x}'|}{c}])}{|\vec{x}-\vec{x}'|}$$

6. Show the following statements.

(a) (7%) In the limit $kr \to \infty$, the expression for $\vec{A}(\vec{x})$ can be recast into

$$\lim_{kr\to\infty}\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}')e^{-ik(\vec{n}\cdot\vec{x}')}d^3\vec{x}' = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_n\frac{(-ik)^n}{n!}\int \vec{J}(\vec{x}')(\vec{n}\cdot\vec{x}')^n d^3\vec{x}',$$

where $\vec{n}$ is a unit vector in the direction of $\vec{x}$

(b) (7%) By applying the result in (a) to the electric dipole radiation field, the electric and magnetic fields take the form in the radiation zone

$$\vec{H} = \frac{ck^2}{4\pi}(\vec{n}\times\vec{p})\frac{e^{ikr}}{r},$$

$$\vec{E} = \sqrt{\frac{\mu_0}{\epsilon_0}}\vec{H}\times\vec{n},$$

where $\vec{p} = \int \vec{x}'\rho(\vec{x}')d^3\vec{x}'$

(c) (6%) Calculate the radiation power per unit solid angle

Hint:

$$\frac{dP}{d\Omega} = \frac{1}{2}\mathrm{Re}[r^2\vec{n}\cdot\vec{E}\times\vec{H}^*].$$