1 (20%)

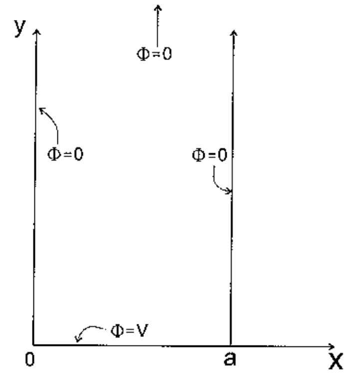
(a) (8%)Solve the Green function for a spherical shell bounded by $r = a$ and $r = b$ $(a < b)$, and it can be expressed as

$$G(\vec{x}, \vec{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y^*_{lm}(\theta', \phi') Y_{lm}(\theta, \phi)}{(2l+1)[1 - (\frac{a}{b})^{2l+1}]} \left(r^l_< - \frac{a^{2l+1}}{r^{l+1}_<} \right) \left(\frac{1}{r^{l+1}_>} - \frac{r^l_>}{b^{2l+1}} \right).$$

(b) (6%)Make use of the result in part(a) to calculate the potential inside the grounded **sphere** with a uniformly charged ring of radius a which is located on the x-y plane, and total charge Q, and b is the radius of the sphere

(c) (6%)Calculate the potential inside a hollow grounded **sphere** with radius b, inside which a uniformly charged rod with length $2b$ and total charge Q is placed on the z axis between the north and south poles of the sphere.

2. (20%)Consider the potential in the region, $0 \le x \le a$, $y \ge 0$ as shown in the figure, subject to the boundary conditions that $\Phi = 0$ at $x = 0$ and $x = a$, while $\Phi = V$ at $y = 0$ for $0 \le x \le a$ and $\Phi \to 0$ for large y.



Show that the potential can be expressed as

$$\Phi(x, y) = \frac{2V}{\pi} \tan^{-1} \left(\frac{\sin \frac{\pi x}{a}}{\sinh \frac{\pi y}{a}} \right).$$

3 (20%)Consider a localized charge distribution $\rho(\mathbf{x})$ that gives rise to an electric field $\mathbf{E}(\mathbf{x})$ throughout space

(a) (8%)Show that the integral can be written as :

$$\int_{r<R} \mathbf{E}(\mathbf{x}) d^3x = -\frac{R^2}{3\epsilon_0} \int d^3x' \frac{r_<}{r^2_>} \mathbf{n}' \rho(\mathbf{x}').$$

(b) (6%)Consider that the sphere of radius R completely encloses the charge density or the charge locates all exterior to the sphere of interest, separately. Verify that

$$\int_{r<R} \mathbf{E}(\mathbf{x}) d^3x = -\frac{\mathbf{p}}{3\epsilon_0},$$

and

$$\int_{r<R} \mathbf{E}(\mathbf{x}) d^3x = -\frac{4\pi}{3} R^3 \mathbf{E}(0);$$

where $\mathbf{p}$ is the electric dipole moment.

(c) (6%)From the results of (b), show that the dipole field should be written as

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \left[\frac{3\mathbf{n}(\mathbf{p}\ \mathbf{n}) - \mathbf{p}}{|\mathbf{x} - \mathbf{x}_0|^3} - \frac{4\pi}{3} \mathbf{p} \delta(\mathbf{x} - \mathbf{x}_0) \right]$$

4 (20%) A current distribution $\vec{J}(\vec{r})$ exists in a medium of unit permeability adjacent to a semi-infinite slab of material having permeability μ and filling the half-space, $z < 0$ (a) Show that for $z > 0$ the magnetic induction can be calculated by replacing the medium of permeability ν by an image current distribution, $\vec{J}^*$, with components,

$$(\frac{\mu-1}{\mu+1})J_x(x,y,-z), \qquad (\frac{\mu-1}{\mu+1})J_y(x,y,-z), \qquad -(\frac{\mu-1}{\mu+1})J_z(x,y,-z)$$

(b) Show that for $z < 0$ the magnetic induction appears to be due to a current distribution

$$(\frac{2\mu}{\mu+1})\vec{J}$$

in a medium of unit permeability.

5 (20%) Suppose the yz plane carries a time-dependent but uniform surface current $K(t)\vec{e}_z$ where $\vec{e}_z$ is the unit vector along z axis (a) Find the electric and magnetic fields at a height x above the plane if

$$K(t) = \begin{cases} 0, & t \le 0, \\ \alpha t, & t > 0. \end{cases}$$

(b) Show that the total power radiated per unit area of surface is

$$\frac{\mu_0 c}{2}[K(t)]^2.$$

Some Useful Formulas.

$$\int_{-1}^{1} P_{l'}^m(x) P_l^m(x) dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{l'l} \qquad Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi}$$

$$\begin{aligned}
as\ \ x << 1 \quad & J_\nu(x) \to \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \\
& N_\nu \to \frac{2}{\pi} \left[\ln\left(\frac{x}{2}\right) + 0.5772 \cdots\right], \quad as\ \ \nu = 0 \\
& N_\nu \to -\frac{\Gamma(\nu)}{\pi} \left(\frac{2}{x}\right)^\nu, \quad as\ \ \nu \neq 0 \\
as\ \ x >> 1, \nu \quad & J_\nu(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \\
& N_\nu(x) \to \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)
\end{aligned}$$

$$\begin{aligned}
& I_\nu(x) = i^{-\nu} J_\nu(ix) \\
& K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) \\
as\ \ x << 1 \quad & I_\nu(x) \to \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \\
& K_\nu \to -\left[\ln\left(\frac{x}{2}\right) + 0.5772 \cdots\right], \quad as\ \ \nu = 0 \\
& K_\nu \to \frac{\Gamma(\nu)}{2} \left(\frac{2}{x}\right)^\nu, \quad as\ \ \nu \neq 0 \\
as\ \ x >> 1, \nu \quad & I_\nu(x) \to \frac{1}{\sqrt{2\pi x}} e^x \left[1 + 0\left(\frac{1}{x}\right)\right] \\
& K_\nu(x) \to \sqrt{\frac{\pi}{2x}} e^{-x} \left[1 + 0\left(\frac{1}{x}\right)\right]
\end{aligned}$$

$$\Omega_{l+1}(x) = \frac{2l}{x}\Omega_l(x) - \Omega_{l-1}(x) \qquad x\Omega_l'(x) = x\Omega_{l-1}(x) - l\Omega_l(x)$$

$$\Omega_l'(x) = \frac{1}{2}(\Omega_{l-1}(x) - \Omega_{l+1}(x)) \qquad x\Omega_l'(x) = l\Omega_l(x) - x\Omega_{l+1}(x)$$

Cylindrical (ρ, ϕ, z)

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$