九十六學年度

# 國 立 中 正 大 學

## 博士班研究生資格考試

所別：物 理 研 究 所

科目：電 動 力 學

（※題目共三張五大題）

96.09.05

1. (a) (10%)Show that the Green function $G(x,y;x',y')$ appropriate for Dirichlet boundary conditions for a square two-dimensional region, $0 \le x \le 1, 0 \le y \le 1$, has an expansion

$$G(x,y;x',y') = 2\sum_{n=1}^{\infty} g_n(y,y')\sin(n\pi x)\sin(n\pi x')$$

where $g_n(y,y')$ satisfies

$$(\frac{\partial^2}{\partial y'^2} - n^2\pi^2)g_n(y,y') = -4\pi\delta(y'-y) \; and \; g_n(y,0) = g_n(y,1) = 0$$

(b) (10%)Taking for $g_n(y,y')$ appropriate linear combinations of $\sinh(n\pi y')$ and $\cosh(n\pi y')$ in the two regions, $y' < y$ and $y' > y$, in accord with the boundary conditions and the discontinuity in slope required by the source delta function,show that the explicit form of $G$ is

$$G(x,y;x',y') = 8\sum_{n=1}^{\infty}\frac{1}{n\sinh(n\pi)}\sin(n\pi x)\sin(n\pi x')\sinh(n\pi y_<)\sinh[n\pi(1-y_>)]$$

where $y_<(y_>)$ is the smaller (larger) of $y$ and $y'$.

2. (10%) Consider a long hollow conducting cylinder of radius $b$ that is divided into equal quarters, alternate segments being held at potential +V and -V. Solve by means of the series solution and show that the potential inside the cylinder is

$$\Phi(\rho,\phi) = \frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{\sin[(4n+2)\phi]}{2n+1}$$

Hint: The series solution:

$$\Phi(\rho,\phi) = a_0 + b_0\ln\rho + \sum_{n=1}^{\infty} a_n\rho^n\sin(n\phi+\alpha_n) + \sum_{n=1}^{\infty} b_n\rho^{-n}\sin(n\phi+\beta_n)$$

3. (20%) Find Green function for potential problems with cylindrical boundary surfaces. The starting point is the equation for Green function:

$$\nabla^2 G(\mathbf{x},\mathbf{x}') = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z') \; .$$

Show that

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{4}{\pi}\int_0^{\infty} dk\cos[k(z-z')]\times$$
$$\left\{\frac{1}{2}I_0(k\rho_<)K_0(k\rho_>) + \sum_{m=1}^{\infty}\cos[m(\phi-\phi')]I_m(k\rho_<)K_m(k\rho_>)\right\}$$

4. (15 points) An infinitely long wire with linear charge density $\lambda$ lies at rest along z–axis, *i.e.* $\rho(\mathbf{r}) = \lambda\delta(x)\delta(y)$.

(a) (3 points) Find the electric field.

(b) (12 points) At $t = 0$ the wire starts to move with constant velocity $v$ in the positive z direction. (i) (2 points) Write down the current density $\mathbf{j}$. (ii) (6 points) Using the formula for the retarded potential

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{c}\int d^3r' \frac{\mathbf{j}(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{|\mathbf{r} - \mathbf{r}'|},$$

to calculate $\mathbf{A}(\mathbf{r}, t)$. (iii) (4 points) Find the $\mathbf{B}$ field.

5. (35 points) A thin linear antenna of length $d$ is excited in such a way that the sinusoidal current makes a full wavelength of oscillation (you may use the current density as $\vec{j}(\vec{x}, t) = I_0 \sin kz\delta(x)\delta(y)e^{-i\omega t}\hat{z}$ and work out this problem in the far zone, *i.e.* $kr \gg 1$).

(a) (10 point) Evaluate the vector potential $\vec{A}(\vec{r})$. (Hint: with the sinusoidal time dependent, the vector potential (spatial part) becomes

$$\vec{A}(\vec{r}) = \frac{1}{c}\int \vec{j}(\vec{r}')\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}d^3r'$$

(b) (5 points) Calculate the magnetic field in this porblem

(c) (10 points) Calculate exactly the power radiated per unit solid angle and plot the angular distribution of radiation

(d) (10 points) Determine the total power radiated and find a numerical value for the radiation resistance

Useful formulae

- Maxwell's equations

$$\nabla \cdot \mathbf{D} = 4\pi\rho, \qquad \nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla\times\mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi\mathbf{J}}{c}$$

$$\mathbf{D} = \epsilon\mathbf{E}, \qquad \mathbf{B} = \mu\mathbf{H}$$

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}$$
$$u = \frac{1}{8\pi}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$$

$$\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} + \mathbf{J}\cdot\mathbf{E} = 0$$

- Using scalar and vector potentials

$$\mathbf{B} = \nabla\times\mathbf{A},$$
$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$\left\{ \begin{array}{c} \phi(\mathbf{r},t) \\ \mathbf{A}(\mathbf{r},t) \end{array} \right\} = \int d^3r' \int dt' \left\{ \begin{array}{c} \rho(\mathbf{r}',t') \\ \mathbf{J}(\mathbf{r}',t') \end{array} \right\} \frac{\delta(t - t' - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{|\mathbf{r}-\mathbf{r}'|}$$

- Simple radiation for the sinusoidal time dependence:

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int \mathbf{J}(\mathbf{r}')\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r',$$
$$\mathbf{B} = \nabla\times\mathbf{A},$$
$$\mathbf{E} = \frac{i}{k}\nabla\times\mathbf{B}.$$

- spherical coordinate:

$$\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta,$$
$$\hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi + \hat{z}(-\sin\theta),$$
$$\hat{\phi} = \hat{x}(-\sin\phi) + \hat{y}\cos\phi.$$