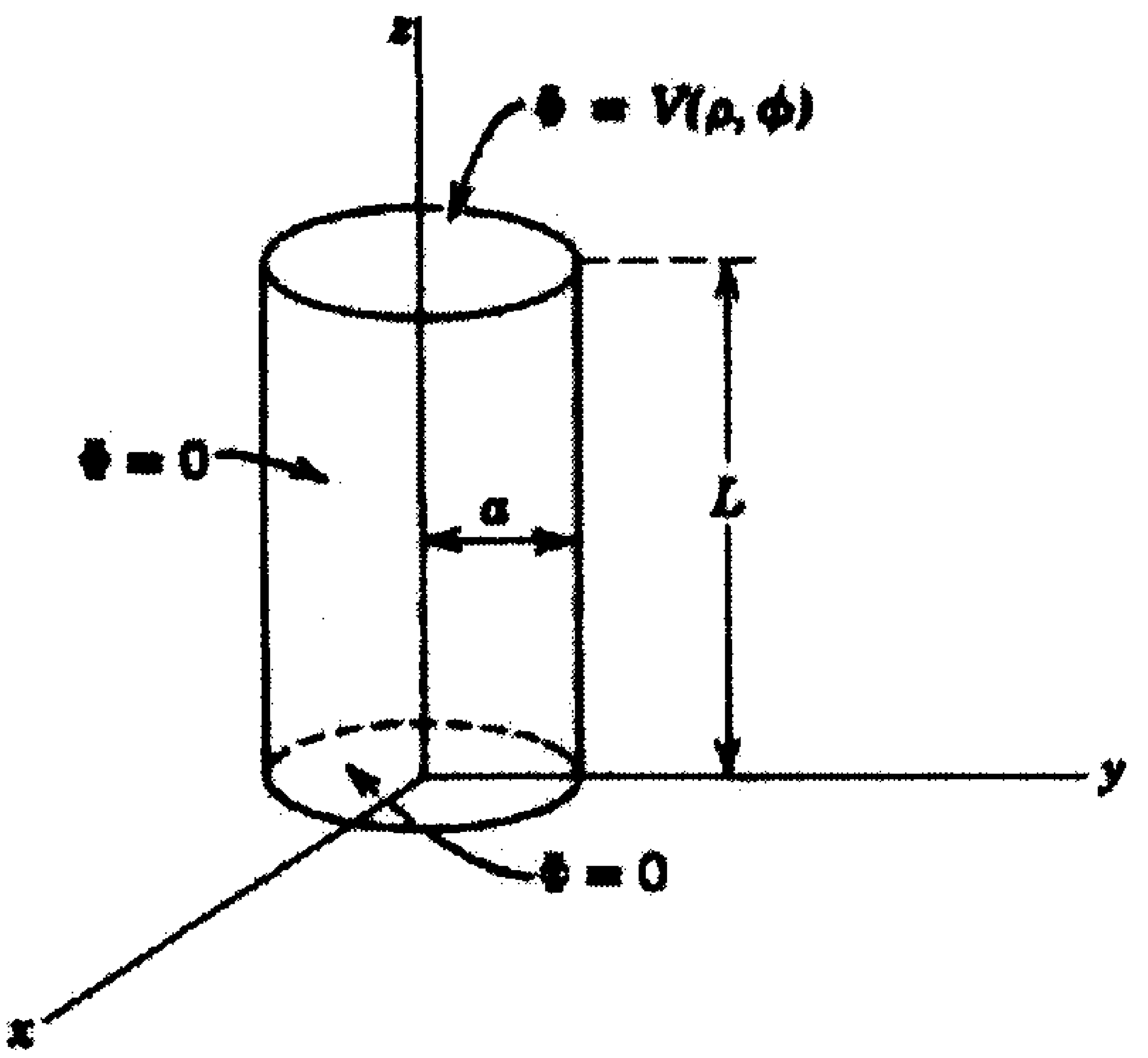
1. (a) (15%)Solve that the Green function for a spherical shell bounded by $r = a$ and $r = b$ can be expressed as

$$G(\vec{x}, \vec{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)}{(2l+1)[1-(\frac{a}{b})^{2l+1}]} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}} \right) \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right)$$

   (b) (7%)Make using of the result in part(a) to calculate the potential inside the grounded sphere with a uniformly charged ring of radius a and total charge Q, and b is the radius of the sphere.

   (c) (8%)Calculate the potential inside the grounded sphere, inside which a uniformly charged rod with length $2b$ and total charge Q is placed.

2. (20%) Consider a cylinder which has a radius a and a hight L, the top and bottom surfaces being at $z = L$ and $z = 0$. The potential on the side and the bottom of the cylinder is zero, while the top has a potential $\Phi = V(\rho, \phi)$.



Please slove the Laplace equation in cylindrical coordinates to derive the general solution for the potential inside is

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho) \sinh(k_{mn}z)(A_{mn} \sin m\phi + B_{mn} \cos m\phi)$$

where $k_{mn} = \frac{x_{mn}}{a}$, and x_{mn} are the roots of $J_m(x_{mn}) = 0$. And find the expression for A_{mn} and B_{mn}.

3. (20 points) A spherical shell of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity ω.

(a) (15 points) Show that the vector potential can be written as

$$\vec{A} = \begin{cases} A_{\text{in}}(r)\, \vec{\omega} \times \vec{r} & r < R \\ A_{\text{out}}(r)\, \vec{\omega} \times \vec{r} & r > R \end{cases}$$

Find the functions $A_{\text{in}}(r)$ and $A_{\text{out}}(r)$.

(b) (5 points) Calculate the magnetic field $\mathbf{B}$ it produces at point outside the shell.

(hint: the easy way to work out the integral is to choose the reference point $\vec{r}$ as the z-axis first, then $\vec{\omega} = \hat{x}\omega \sin\psi + \hat{z}\omega \cos\psi$, with ψ is the angle between $\vec{r}$ and $\vec{\omega}$.)

4. (30 points) A circular wire antenna of radius a is exactly one wave length long at the angular frequency ω $(\omega a/c = 1)$. The current density flowing in the wire is

$$\vec{\mathbf{J}}(\vec{\mathbf{r}}, t) = \hat{\phi} I \cos\phi \cos(\omega t) \delta(r - a) \delta(\cos\theta)/a \; .$$

(a) Calculate the potential $\vec{\mathbf{A}}$ in the radiation zone $\omega r/c \gg 1$. (The angular integration over ϕ should be evaluated in terms of the Bessel functions J_n and the derivatives J_n'.)

(b) Use these potentials to calculate the electric and magnetic fields in the radiation zone. The fields should be written in spherical coordinates so that the orthogonality relations $E_r = B_r = 0$, $E_\theta = B_\phi$, and $E_\phi = -B_\theta$ are shown explicitly.

(c) Finally calculate the time–averaged power radiated per unit solid angle $dP/d\Omega$.

Useful formulae

- Cylindrical coordinate:

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$$

- The Modified Bessel equation:

$$\frac{1}{x}\frac{d}{dx}x\frac{df(x)}{dx} - (1+\frac{\nu^2}{x^2})f(x) = 0$$

has two linearly independent solutions, $I_\nu(x)$ and $K_\nu(x)$.

- Bessel equation/function:

$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}J_n(k\rho) - \frac{n^2}{\rho^2}J_n(k\rho) = -k^2 J_n(k\rho)$$

$$\int_0^a \left[J_n\left(\alpha_{nm}\frac{\rho}{a}\right)\right]^2 \rho d\rho = \frac{a^2}{2}[J_{n+1}(\alpha_{nm})]^2$$

$$\int_0^1 xJ_0(ax)dx = \frac{J_1(a)}{a} ,$$

$$J_n(x) = \frac{1}{2\pi i^n}\int_0^{2\pi} d\phi e^{ix\cos\phi}\left[\cos n\phi - i\sin n\phi\right] ,$$

$$\tfrac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x) ,$$

$$2\tfrac{dJ_n(x)}{dx} = J_{n-1}(x) - J_{n+1}(x) .$$

- electric field for a dipole $\mathbf{p}$ at $\mathbf{r}_0$:

$$\mathbf{E}(\mathbf{r}) = \frac{3\mathbf{n}(\mathbf{p}\cdot\mathbf{n}) - \mathbf{p}}{|\mathbf{r}-\mathbf{r}_0|^3} - \frac{4\pi}{3}\mathbf{p}\delta(\mathbf{r}-\mathbf{r}_0) .$$

- magnetic field for a dipole $\mathbf{m}$ at origin:

$$\mathbf{B}(\mathbf{r}) = \frac{3\mathbf{n}(\mathbf{m}\cdot\mathbf{n}) - \mathbf{m}}{|\mathbf{r}|^3} + \frac{8\pi}{3}\mathbf{m}\delta(\mathbf{r}) .$$

- electric dipole moment:

$$\mathbf{p} = \int \mathbf{r}'\rho(\mathbf{r}')d^3r' .$$

- electric traceless quadrupole moment:

$$Q_{ij} = \int (3x_i'x_j' - r'^2\delta_{ij})\rho(\mathbf{r}')d^3r' .$$

- Maxwell's equations

$$\nabla\cdot\mathbf{D} = 4\pi\rho, \qquad \nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = 0$$

$$\nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{H} - \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} = \frac{4\pi\mathbf{J}}{c}$$

$$\mathbf{D} = \epsilon\mathbf{E}, \qquad \mathbf{B} = \mu\mathbf{H}$$

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}$$

$$u = \frac{1}{8\pi}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$$

$$\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} + \mathbf{J}\cdot\mathbf{E} = 0 .$$

- Using scalar and vector potentials

$$\mathbf{B} = \nabla\times\mathbf{A} ,$$

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} .$$

$$\left\{ \begin{matrix}\phi(\mathbf{r},t)\\ \mathbf{A}(\mathbf{r},t)\end{matrix} \right\} = \int d^3r' \int dt' \left\{ \begin{matrix}\rho(\mathbf{r}',t')\\ \mathbf{J}(\mathbf{r}',t')\end{matrix} \right\} \frac{\delta(t - t' - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{|\mathbf{r}-\mathbf{r}'|}$$

- Simple radiation for the sinusoidal time dependence:

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int \mathbf{J}(\mathbf{r}')\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r' ,$$

$$\mathbf{B} = \nabla\times\mathbf{A} ,$$

$$\mathbf{E} = \frac{i}{k}\nabla\times\mathbf{B} .$$

- spherical coordinate:

$$\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta ,$$

$$\hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi + \hat{z}(-\sin\theta) ,$$

$$\hat{\phi} = \hat{x}(-\sin\phi) + \hat{y}\cos\phi .$$