1. (a) (5%)Show that the Green function $G(x,y;x',y')$ appropriate for Dirichlet boundary conditions for a square two-dimensional region, $0 \le x \le 1, 0 \le y \le 1$, has an expansion

$$G(x,y;x',y') = 2\sum_{n=1}^{\infty} g_n(y,y') \sin(n\pi x)\sin(n\pi x')$$

where $g_n(y,y')$ satisfies

$$(\frac{\partial^2}{\partial y'^2} - n^2\pi^2)g_n(y,y') = -4\pi\delta(y'-y) \ and \ g_n(y,0) = g_n(y,1) = 0$$

(b) (10%)Taking for $g_n(y,y')$ appropriate linear combinations of $\sinh(n\pi y')$ and $\cosh(n\pi y')$ in the two regions, $y' < y$ and $y' > y$, in accord with the boundary conditions and the discontinuity in slope required by the source delta function,show that the explicit form of G is

$$G(x,y;x',y') = 8\sum_{n=1}^{\infty} \frac{1}{n\sinh(n\pi)} \sin(n\pi x)\sin(n\pi x')\sinh(n\pi y_<)\sinh[n\pi(1-y_>)]$$

where $y_<(y_>)$ is the smaller (larger) of y and y'.

2. (20%)Solve that the Green function for a spherical shell bounded by $r = a$ and $r = b$ can be expressed as

$$G(\vec{x},\vec{x}') = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)[1-(\frac{a}{b})^{2l+1}]} \left(r^l_< - \frac{a^{2l+1}}{r^{l+1}_<}\right)\left(\frac{1}{r^{l+1}_>} - \frac{r^l_>}{b^{2l+1}}\right)$$

3. (15%)Solve that the Green function for the infinite system in terms of the cylindrical coordinates can be expressed as

$$G(\vec{x},\vec{x}') = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\ e^{im(\phi-\phi')}\cos[k(z-z')]I_m(k\rho_<)K_m(k\rho_>)$$

4.(20%) Point charge in the presence of a grounded conducting sphere —- A point charge q is located at $\vec{y}$ relative to the origin, around which is centered a grounded conducting sphere of radius a as illustrated in Fig. 2.2.

(a) By the method of images, find the potential $\Phi(\vec{x})$ outside the sphere such that $\Phi(|\vec{x}| = a) = 0$.

(b) What is the charge density σ induced on the surface of the sphere? Plot the surface-charge density σ as a function of γ, the angle between $\vec{x}$ and $\vec{y}$.

(c) Calculate the total force acting on the charge q. Give the force on each element of area da of the conducting sphere.

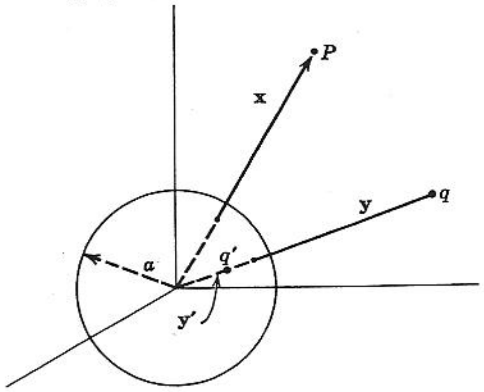

Figure 2.2 Conducting sphere of radius a, with charge q and image charge q'.

5.(20%) Derivation of Green's theorem — Consider a well-behaved vector field $\vec{A}$ defined in the volume V bounded by the closed surface S. The divergence theorem states:

$$\int_V \nabla \cdot \vec{A}\, d^3x = \oint_S \vec{A} \cdot \vec{n}\, da\,.$$

(a) Derive the Green's theorem for the choice $\vec{A} = \phi \nabla \psi$, where ϕ and ψ are arbitrary scalar fields.

(b) In part (a), we set $\phi = \Phi$ with $\nabla^2 \Phi = -\rho/\epsilon_0$ and $\psi = G(\vec{x}, \vec{x'})$ with the specified properties of the Green's function G

$$\nabla'^2 G(\vec{x}, \vec{x'}) = -4\pi \delta(\vec{x} - \vec{x'})\,.$$

Find the general solution of $\Phi(\vec{x})$ for Dirichlet boundary conditions, *i.t.*, $G_D(\vec{x}, \vec{x'}) = 0$ for $\vec{x'}$ on S.

(c) Alternatively, if we are seeking the general solution of $\Phi(\vec{x})$ for Neumann boundary conditions, what is the simplest allowable boundary condition on the Green's function G_N? Write down the general solution of $\Phi(\vec{x})$ in this case.

6.(10%) Green function for the sphere; general solution for the potential —- Using the result of part (b) in problem 5, find the solution of the Laplace equation outside a sphere of radius a with the potential specified on its surface, that is, $\Phi(a, \theta', \phi')$ is known in spherical coordinates. Note that the appropriate Green's function is given by

$$G_D(\vec{x}, \vec{x'}) = \frac{1}{\left|\vec{x} - \vec{x'}\right|} - \frac{a}{x'\left|\vec{x} - \frac{a^2}{x'^2}\vec{x'}\right|}\,.$$