1. ( 10 points ) Find the eigenvalues and normalized eigenvectors of the matrix

$$\begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. Let $\lambda_i \, (i=1,2,3)$ be the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 2 \\ -1 & 2 & -2 \end{pmatrix}$$

Calculate the sums

(a) ( 10 points ) $\sum_{i=1}^{3} \lambda_i$ and (b) ( 10 points ) $\sum_{i=1}^{3} \lambda_i^2$ .

3. Solve the following first-order equations for the boundary conditions given :

(a) ( 10 points ) $\frac{dy}{dx} - \frac{y}{x} = 1, \quad y(1) = 1$ ;

(b) ( 10 points ) $\frac{dy}{dx} - y \tan x = 1, \quad y(0) = 2.$

4. (15 points) The spherical polar unit vectors are expressed in Cartesian unit vectors as

$$\begin{aligned} \hat{r} &= \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta\,, \\ \hat{\theta} &= \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta\,, \\ \hat{\phi} &= -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi\,. \end{aligned}$$

Resolve the Cartesian unit vectors into their spherical polar components.

5. (15 points) Find the general solution of the given differential equation by transforming it into an equation with constant coefficients by a change of independent variable: $t \to x = u(t)$,

$$y''(t) + ty'(t) + e^{-t^2}y(t) = 0\,, \qquad -\infty < t < \infty\,.$$

6. (20 points) Consider the linear second-order ordinary differential equation

$$x(1-x)y''(x) + [c - (1+2a)x]y'(x) - a^2y(x) = 0\,,$$

where $a$ and $c$ are constants and $c \neq 0, -1, -2, -3, \ldots$.

(a) (15 points) Show that $x = 0$ is a regular singular point. Find a series solution of the equation about $x = 0$ and determine the range of convergence.

(b) (5 points) Show that $x = \infty$ is also a regular singular point.