CALCULUS

1. True or False? If true, give reasons; if false, give a counterexample.

(1) If $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a}(f(x)+g(x))$ does not exist. (5%)

(2) If f is continuous at a, then so is $|f|$. (5%)

(3) If $f(x) < g(x)$ for all $x \neq a$ and if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$. (5%)

2. Let $f(x) = x + 2x^2 \sin(\frac{1}{x})$ when $x \neq 0$ and $f(0) = 0$.

(1) Find $f'(0)$. (5%)

(2) Show that f can not be increasing on any interval containing $x = 0$. (5%)

3. The rate at which a tank drains is proportional to the square root of the depth of liquid in the tank above the level of the drain: If $V(t)$ is the volume of liquid in the tank at time t and $y(t)$ is the height of the surface of the liquid above the drain, then $\frac{dV}{dt} = -k\sqrt{y}$, where k is a constant depending on the size of the drain. For a cylinderical tank with constant cross-sectional area A with drain at the bottom:

(1) Verify that the depth $y(t)$ of liquid in the tank at time t satisfies $\frac{dy}{dt} = -(\frac{k}{A})\sqrt{y}$. (5%)

(2) Verify that if the depth of liquid in the tank at $t = 0$ is y_0, then the depth at subsequent times during the draining process is $y = (\sqrt{y_0} - \frac{kt}{2A})^2$. (5%)

(3) If the tank drains completely in time T, express the depth $y(t)$ at time t in terms of y_0 and T. (5%)

4. Apply Newton's method to the function $f(x) = \sqrt{x}$ when $x \geq 0$, $f(x) = \sqrt{-x}$ when $x < 0$, starting with the initial quess $x_0 = a > 0$. Calculate x_1 and x_2. What happens? (Make a skech.) If you ever observed this behaviour when you were using Newton's method to find a root of an equation, what would you do next. (10%)

5. (10%) Test for the converences of the following improper integrals.

(1) $\int_0^\infty \frac{x^2 dx}{1+x^3}$ (2) $\int_{-\infty}^{\infty} (x^2 \sin^2 x) e^{-x^4} dx$

6.(10%) Find the volume of the region in R^3 defined by the inequalities: $x^2 + y^2 \leq 1$ and $y^2 + z^2 \leq 1$.

7.(10%) Evalute the improper integral $\int_0^\infty x^2 e^{-x^2} dx$.

8.(10%) Prove the following identity:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots + \frac{(-1)^n}{2n+1} + \ldots\ldots = \frac{\pi}{4}.$$

9.(10%) Let C be the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in counterclockwise direction. Evalute the following line integral:

$$\int_C 3xdy + 4ydx.$$